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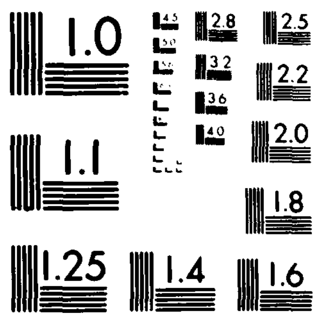

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Consider the random uniform placement of a finite number of arcs on the circle, where the arc lengths are sampled from a distribution on (0,1). We provide exact formulae for the probability that the circle is completely covered and for the distribution of the number of uncovered gaps, extending Stevens' (1939) formulae for the case of fixed equal arc lengths. A special class of arc length distributions is considered, and exact probabilities of coverage are tabulated for the uniform distribution on (0,1). Some asymptotic results for the number of gaps are also given.

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COVERING THE CIRCLE WITH RANDOM ARCS OF RANDOM SIZES

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key words: Geometrical Probability, Coverage Probability, Spacings.

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ABSTRACT

Consider the random uniform placement of a finite number of arcs on the circle, where the arc lengths are sampled from a distribution on (0,1). We provide exact formulae for the probability that the circle is completely covered and for the distribution of the number of uncovered gaps, extending Stevens' (1939) formulae for the case of fixed equal arc lengths. A special class of arc length distributions is considered, and exact probabilities of coverage are tabulated for the uniform distribution on (0,1). Some asymptotic results for the number of gaps are also given.

1. Introduction

Many authors have considered problems relating to the coverage of a circle by random arcs of equal sizes, including Whitworth (1897), Stevens (1939), Fisher (1940), Votaw (1946), Domb (1947), Flatto and Konheim (1962), Solomon (1978), Holst (1980 a,b,c and 1981), and Siegel (1978b and 1979). When the arc lengths are not equal, few results are known. Conditions under which a given infinite sequence of arc lengths will almost surely cover the circle were studied by Dvoretzki (1956), Mandelbrot (1972), Shepp (1972), and others. However, when the number of arcs is finite, there is no known simple formula for the coverage probability (Shepp, 1972), although some results are available concerning the amount of the circle covered in this case (Yadin and Zachs, 1980). Some approximations to the coverage probability using moments and simulations were provided by Siegel (1978a).

We provide an exact formula for the probability of covering the circle with n random arcs of random independent sizes in Section 2. The formula is derived by considering the number of uncovered gaps, thereby extending Stevens' (1939) formula to the case of random arc length. Feasibility of computation is demonstrated in Section 3 for the special case of arcs of uniform lengths and a related class of distributions. Some asymptotics concerning the number of uncovered gaps are obtained in Section 4.

2. Distribution of the Number of Gaps and the Probability of Coverage

Consider n arcs placed uniformly and independently at random on a circle of circumference one. Let the arc lengths be independent and identically distributed random variables $L_1, \ldots, L_n$, chosen from the cumulative distribution function F on $(0,1)$. The number of uncovered gaps will be denoted N_n. Note that $N_n = 0$ is the event of complete coverage. An example with $N_4 = 2$ is illustrated in the figure.

Theorem 2.1. The distribution of N_n is given by

$$P(N_n = m) = \binom{n}{m} \sum_{k=m}^{n} \binom{n-m}{k-m} (-1)^{k-m} \xi_k \tag{2.1}$$

where

$$\xi_k = \int_{(\sum_{i=1}^{k} u_i = 1)} \left[\prod_{i=1}^{k} F(u_i)\right] \left[\sum_{j=1}^{k} \int_0^{u_j} F(v)dv\right]^{n-k} d\underset{\sim}{u} \tag{2.2}$$

$$= E\left\{\left[\prod_{i=1}^{k} F(S_{ik})\right] \left[\sum_{j=1}^{k} E_L(S_{jk}-L)_+\right]^{n-k}\right\} \tag{2.3}$$

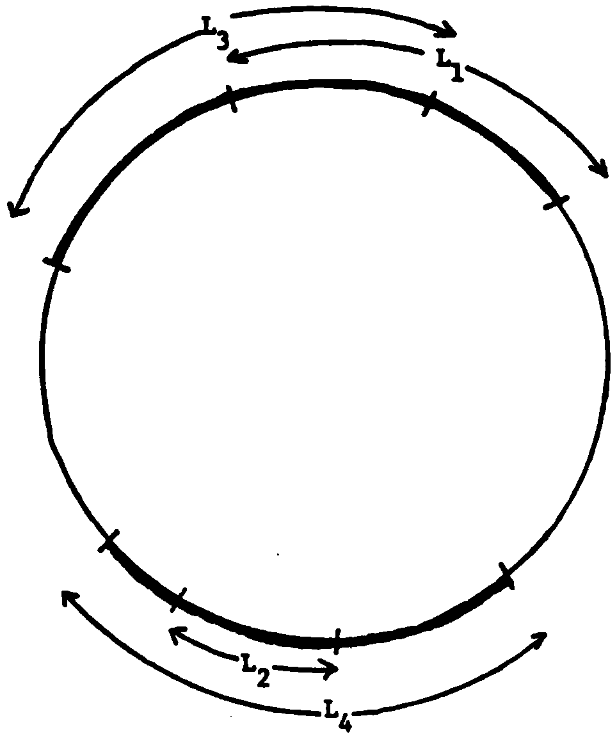
and $S_{1k}, \ldots, S_{kk}$ denotes the spacings between k independent uniform points on the circle. The positive part function is $(t)_+ = \max(t,0)$. The first terms are

$$\xi_0 = 1$$

$$\xi_1 = [1 - E(L)]^{n-1}$$

$$\xi_2 = \int_0^1 F(u)F(1-u) \left[\int_0^u F(v)dv + \int_0^{1-u} F(v)dv\right]^{n-2} du$$

and the first moment is $E(N_n) = n[1 - E(L)]^{n-1}$.



<u>Figure</u>. The event of $N_4 = 2$ uncovered gaps when $n=4$ random arcs are placed on the circle.

<u>Corollary</u>. The probability of completely covering the circle is

$$P(N_n=0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_{(\sum_{i=1}^{k} u_i=1)} \left[\prod_{i=1}^{k} F(u_i)\right] \left[\sum_{j=1}^{k} \int_0^{u_j} F(v)dv\right]^{n-k} d\underline{u}. \quad (2.4)$$

The first m terms suffice whenever arcs must have length at least $1/m$ because the product term will be zero for the higher terms.

<u>Proof</u>. Consider the events A_i = "clockwise endpoint of arc i is not covered". The figure shows the case in which A_1 and A_4 occur. Define $\xi_k = P(\bigcap_{i=1}^{k} A_i)$. Using reasoning analogous to Stevens (1939, formula 4.62) we obtain (2.1). To calculate ξ_k, observe that

$$\xi_k = P\left(\left[\bigcap_{i=1}^{k} \{L_i \le S_{ik}\}\right] \cap B_{n-k}\right) \tag{2.5}$$

where $S_{1k}, \ldots, S_{kk}$ are the lengths of the gaps between clockwise endpoints of the first k gaps and B_{n-k} is the event "none of the arcs $k+1, \ldots, n$ cover the clockwise endpoints of arcs $1, \ldots, k$." Conditioning on $S_{1k}, \ldots, S_{kk}$, independence of the arc lengths gives us

$$\xi_k = E\left\{P(B_{n-k}|S_{1k}, \ldots, S_{kk}) \prod_{i=1}^{k} F(S_{ik})\right\} . \tag{2.6}$$

An additional arc will fall within one of the first k gaps with conditional probability

$$E\left\{\sum_{j=1}^{k} (S_{jk}-L)_+ | S_{1k}, \ldots, S_{kk}\right\} .$$

Again using independence of arc lengths, we have

$$P(B_{n-k}|S_{1k}, \ldots, S_{kk}) = \left[\sum_{j=1}^{k} E_L(S_{jk}-L)_+\right]^{n-k} \tag{2.7}$$

which, together with (2.6) establishes (2.3). Writing expectations as integrals and integrating by parts we obtain (2.2). Because N_n is the sum of indicator functions $N_n = \sum_{i=1}^{n} I(A_i)$, its first moment is

$$E(N_n) = nP(A_1) = n\xi_1 = n[1 - E(L)]^{n-1} . \quad \square$$

The formula (2.4) for coverage probability does reduce to Stevens' (1939) formula when arcs have fixed length a. In this case $F(x) = I(x \ge a)$ and from (2.3) we have

$$\xi_k = E\left\{\left[\prod_{i=1}^{k} I(S_{ik} \geq a)\right]\left[\sum_{j=1}^{k} E(S_{jk}-a)_+\right]^{n-k}\right\}.$$

The product term allows us to drop the positive part to obtain

$$\xi_k = [P(\text{all } S_{ik} \geq a)] \cdot (1-ka)^{n-k}$$

which reduces to

$$\xi_k = (1-ka)_+^{n-1} . \tag{2.8}$$

Substituting this into formula (2.4) we obtain Stevens' formula for equal arcs:

$$P(\text{cover}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1-ka)_+^{n-1} .$$

3. Uniform Arc Length and Related Distributions

If the cumulative distribution function of the arc length has the special form

$$F(x) = x^{\alpha} , \quad 0 < x < 1 \tag{3.1}$$

for some $\alpha > 0$, then the probabilities in Section 2 can be simplified by performing the integrations.

Theorem 3.1. If $F(x) = x^{\alpha}$, then ξ_k from Theorem 2.1 is

$$\xi_k = \frac{(k-1)!}{(\alpha+1)^{n-k}\,\Gamma[(\alpha+1)n]} \sum_{(\sum_{i=1}^{k} m_i = n-k)} \binom{n-k}{m_1,\ldots,m_k} \prod_{i=1}^{k} \Gamma[(\alpha+1)(m_i+1)] \tag{3.2}$$

where the sum is over nonnegative integers m_i. An equivalent formula with fewer terms to sum is obtained if we define $\nu_j = \#(m_i = j)$:

$$\xi_k = \frac{(k-1)!}{(\alpha+1)^{n-k}\,\Gamma[(\alpha+1)n]} \sum_{\substack{m_1 \le \cdots \le m_k \\ \Sigma m_i = n-k}} \binom{k}{\nu_0, \ldots, \nu_k} \binom{n-k}{m_1, \ldots, m_k} \tag{3.3}$$

$$\cdot \prod_{i=1}^{k} \Gamma[(\alpha+1)(m_i+1)].$$

Corollary. The probability of covering the circle with n random arcs whose lengths are chosen independently from the uniform distribution on $(0,1)$ is

$$P(\text{cover}) = 1 + \frac{n!}{(2n-1)!} \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)!\,k\,2^{n-k}} \sum_{(\sum_{i=1}^{k} m_i = n-k)} \binom{n-k}{m_1, \ldots, m_k} \tag{3.4}$$

$$\cdot \prod_{i=1}^{k} (2m_i+1)!\ .$$

Despite its complicated appearance, it was tabulated on a hand calculator for up to 10 arcs, as shown in the table, second column. The third column allows comparison with the case of fixed arcs of the expected length, 1/2. We see that the differences in coverage probability are most evident when n is small.

To further assess the differences between placing fixed and random arc lengths, column 4 of the table lists the fixed arc length whose coverage probability is equal to that of the uniform random arc lengths. These equivalent fixed arc length values converge to .5, again suggesting the similarity as n increases between the random and fixed arc length coverage probabilities.

Table.

n	Coverage Probabilities: random arc lengths $\mathcal{U}(0,1)$	Coverage Probabilities: fixed arc lengths .5	fixed arc lengths needed to achieve coverage probability listed for $\mathcal{U}(0,1)$ arcs
1	0	0	-
2	.167	0	.583
3	.383	.250	.547
4	.569	.500	.524
5	.730	.688	.518
6	.834	.813	.512
7	.901	.891	.508
8	.943	.938	.506
9	.967	.965	.505
10	.982	.980	.504

Proof of Theorem 3.1. Substituting $F(x) = x^{\alpha}$ into (2.3) and performing the inner expectation, we have

$$\xi_k = E\left\{\left(\prod_{i=1}^{k} S_{ik}^{\alpha}\right)\left(\sum_{j=1}^{k} \frac{S_{jk}^{\alpha+1}}{\alpha+1}\right)^{n-k}\right\}.$$

Distributions are preserved by the representation $S_{ik} = X_i/(\sum_{j=1}^{k} X_j)$ where $X_1, \ldots, X_n$ are independent exponential random variables with mean 1. Moreover, $(S_{1k}, \ldots, S_{kk})$ is independent of $\sum_{j=1}^{k} X_j$, so we can show that

$$\xi_k = \frac{E\left[\left(\prod_{i=1}^{k} X_i^{\alpha}\right)\left(\sum_{j=1}^{k} \frac{X_j^{\alpha+1}}{\alpha+1}\right)^{n-k}\right]}{E\left(\sum_{j=1}^{k} X_j\right)^{(\alpha+1)n-k}}. \tag{3.5}$$

The distribution of $\sum_{j=1}^{k} X_j$ is Gamma(k,1) so the denominator of (3.5) is

$$\frac{\Gamma[(\alpha+1)n]}{(k-1)!} \, . \tag{3.6}$$

The numerator of (3.5) can be written as

$$\frac{[\Gamma(\alpha+1)]^k}{(\alpha+1)^{n-k}} E\left(\sum_{i=1}^{k} Y_i^{\alpha+1}\right)^{n-k} \tag{3.7}$$

where $Y_1, \ldots, Y_k$ are independent Gamma $(\alpha+1,1)$. We do a multinominal expansion to obtain

$$\frac{[\Gamma(\alpha+1)]^k}{(\alpha+1)^{n-k}} \sum_{(\sum_{i=1}^{k} m_i = n-k)} \binom{n-k}{m_1,\ldots,m_k} \prod_{i=1}^{k} E(Y_i^{\alpha+1})^{m_i} \, . \tag{3.8}$$

The expectation can be evaluated:

$$E(Y_i^{\alpha+1})^{m_i} = \Gamma\,[(\alpha+1)(m_i+1)]/\Gamma(\alpha+1) \tag{3.9}$$

Substituting (3.9) in (3.8), substituting the resulting numerator and the denominator (3.6) for the ratio in (3.5), we have the result (3.2). Formula (3.3) is obtained by combining summands with the same set of exponents $\{m_1, \ldots, m_k\}$ and the same multiplicities. □

<u>REMARK</u>: In a similar way, distributions of the form $F(x) = 1-(1-x)^n$, $o<x<1$, $n=1,2,\ldots$ can be treated.

4. Asymptotic Results

Consider the situation of Section 2, but let the arc lengths be chosen from the distributions F_n which tend to zero in a suitable way when $n \to \infty$.

Theorem 4.1. Suppose that the arc lengths $0 \leq L_{1n}, L_{2n}, \ldots, L_{nn} \leq 1$ are independent, identically distributed, random variables with the distribution F_n such that

$$\text{(A)} \qquad E(N_n) = n\,(1-\mu_n)^{n-1} \to \lambda\ ,\ n \to \infty \tag{4.1}$$

where $\mu_n = E(L_{jn})$, and for each k=2,3,... we have

$$\text{(B)} \qquad \frac{\int_{-\infty}^{\infty} [E\{F_n(X/k)\exp(n\int_{X/k}^{\infty}(1-F_n(\ell))d\ell/(1-k\mu_n) + iu(X-1))\}\,]^k du}{2\pi k^k e^{-k} / k!}$$

$$\to 1,\quad n \to \infty\ , \tag{4.2}$$

where X is exponential with mean 1. Then

$$N_n \xrightarrow{D} \text{Poisson}\ (\lambda),\ n \to \infty\ . \tag{4.3}$$

Proof. By the Lemma below it follows that we need only establish that for all k=1,2,3,...,

$$n^k \xi_k \to \lambda^k\ ,\ n \to \infty\ , \tag{4.4}$$

where ξ_k was defined in Theorem 2.1. By the condition (A) this is true for k=1. Using the identity $(t)_+ = t + (-t)_+$, we see that

$$n^k\xi_k = n^k E_S\left\{(1-k\mu_n+\sum_{j=1}^{n} E_L(L_{jn}-S_{jk})_+)^{n-k}\prod_{j=1}^{k} F_n(S_{jk})\right\}$$

$$= n^k\ (1-k\mu_n)^{n-k}\ \cdot \tag{4.5}$$

$$\cdot\ E_S\left\{(1+\sum_{j=1}^{k} E_L(L_{jn}-S_{jk})_+/(1-k\mu_n))^{n-k}\prod_{j=1}^{k} F_n(S_{jk})\right\}.$$

By (4.1) it follows that

$$n^k(1-k\mu_n)^{n-k} \to \lambda^k,\ n \to \infty. \tag{4.6}$$

Hence (4.4) is true if for each k=2,3,...

$$E_S\left\{(1+Y_n)^{n-k}\prod_{j=1}^{k} F_n(S_{jk})\right\} \to 1,\ n \to \infty\ , \tag{4.7}$$

where we have introduced the random variable

$$Y_n = \sum_{j=1}^{k} E_L(L_{jn}-S_{jk})_+/(1-k\mu_n) = \sum_{j=1}^{k} \int_{S_{jk}}^{1} (1-F_n(\ell))d\ell/(1-k\mu_n). \tag{4.8}$$

As $1+x \le e^x$ for $x \ge o$ it follows that (4.7) holds if

$$E_S\left\{e^{nY_n}\prod_{j=1}^{k} F_n(S_{jk})\right\} \to 1,\ n \to \infty\ . \tag{4.9}$$

In a similar way as in Holst (1981) Lemmas 2.1 and 3.1 one can prove that

$$E_S\left\{e^{nY_n}\prod_{j=1}^{k} F_n(S_{jk})\right\} = (2\pi\ k^k e^{-k}/k!)^{-1}\ \cdot$$

$$\cdot\int_{-\infty}^{\infty} [E\left\{F_n(X/k)\ \exp(n\int_{X/k}^{\infty}(1-F_n(\ell))d\ell/(1-k\mu_n)+iu(X-1))\right\}]^k du \tag{4.10}$$

Combining (4.9) and (4.10) with the Assumption (B) proves the assertion. □

<u>Remark</u>. From the proof it actually follows that the probability generating function of N_n converges to that of a Poisson(λ), provided (4.1) and (4.2) are fulfilled.

<u>Corollary</u>. If the arcs have equal lengths μ_n and $E(N_n) \to \lambda$, then

$$N_n \xrightarrow{D} \text{Poisson } (\lambda), \ n \to \infty .$$

<u>Proof</u>. In this case the integral in (4.2) simplifies to

$$\int_{-\infty}^{\infty}[\int_{k\mu_n}^{\infty}\exp\ (iu(x-1)-x)dx]^k du =$$

$$= \int_{-\infty}^{\infty} [\exp\{-iu-k\mu_n+iuk\mu_n\}/(1-iu)]^k du$$

$$\to \int_{-\infty}^{\infty} [e^{-iu}/(1-iu)]^k du = 2\pi\ k^k e^{-k}/k! \ , \ n \to \infty ,$$

As $(1-iu)^{-k}$ is the characteristic function of a Gamma (k,1) distribution. $\square$

In a similar way by explicit calculation of the integrand in (4.2) one obtains:

<u>Corollary</u>. Let the arc lengths take the values $o<a_1\mu_n< \dots < a_r\ \mu_n$ with probabilities $p_1, \dots, p_r$ where $p_1+ \dots + p_r = 1$ and $a_1p_1+ \dots + a_rp_r = 1$. If $\mu_n \to o$ such that $E(N_n) \to \lambda$, then

$$N_n \xrightarrow{D} \text{Poisson } (\lambda), \ n \to \infty .$$

Lemma. Let $(I_{1n}, \ldots, I_{nn})$, $n=1, 2, \ldots$ be a triangular array of exchangeable random variables that take only the values 0 or 1. Suppose there exists a $\lambda > 0$ such that for every fixed $k=1, 2, \ldots$ we have

$$n^k E\left(\prod_{i=1}^{k} I_{in}\right) \to \lambda^k \quad \text{as} \quad n\to\infty . \tag{4.11}$$

Then

$$\sum_{i=1}^{n} I_{in} \xrightarrow{D} \text{Poisson } (\lambda) \quad \text{as} \quad n \to \infty . \tag{4.12}$$

Proof. Consider the Stirling number of the second kind $S(m,k)$ = "number of partitions of m distinct elements into k sets." By exchangeability it follows that the m^{th} moment of the left hand side of (4.11) is

$$E\left(\sum_{i=1}^{n} I_{in}\right)^m = \sum_{k=1}^{m} S(m,k) \frac{n!}{(n-k)!} E\left[\prod_{i=1}^{k} I_{in}\right]$$

which tends to the m^{th} moment

$$\sum_{k=1}^{m} S(m,k) \lambda^k \tag{4.13}$$

of a Poisson (λ) random variable as $n \to \infty$, establishing (4.12) by convergence of moments. □

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